26. Fourier Transform: Examples

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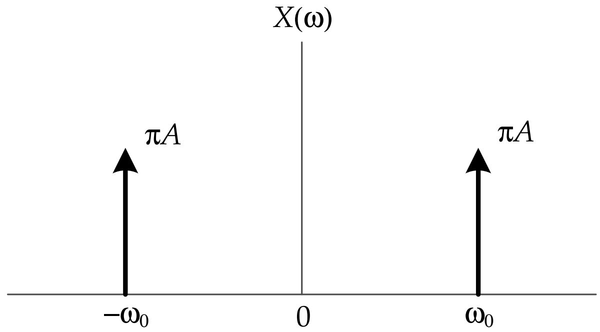


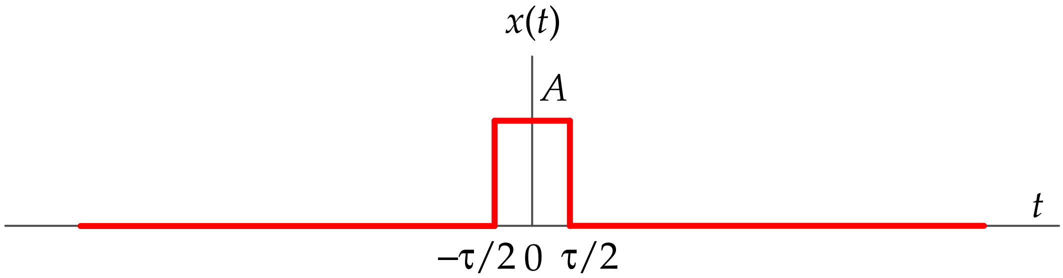
Fourier transform, denote by $X(\omega)$, of the signal $x(t)$, periodic or aperiodic, is given by the integral,

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

If you were given the Fourier transform $X(\omega)$, you can figure out the original signal $x(t)$ using the inverse operation, called inverse Fourier transform, given by the integral,

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Q1. For the following signal $x(t) = A\,\mathrm{rect}\left(\frac{t}{\tau}\right)$, determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q1. Solution. Using Fourier transform integral

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{-\tau/2} 0\,\left(e^{-j\omega t}\right) dt + \int_{-\tau/2}^{\tau/2} A\left(e^{-j\omega t}\right) dt + \int_{\tau/2}^{\infty} 0\,\left(e^{-j\omega t}\right) dt$$

$$X(\omega) = A\left[\frac{1}{-j\omega} e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$X(\omega) = \frac{A}{-j\omega}\left[e^{\frac{-j\omega\tau}{2}} - e^{\frac{+j\omega\tau}{2}}\right] = \frac{2A}{\omega}\left[\frac{e^{\frac{+j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}}{2j}\right]$$

$$X(\omega) = \frac{2A}{\omega}\left[\frac{e^{\frac{+j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}}{2j}\right] = 2A \times \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = A\tau \times \frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}}$$

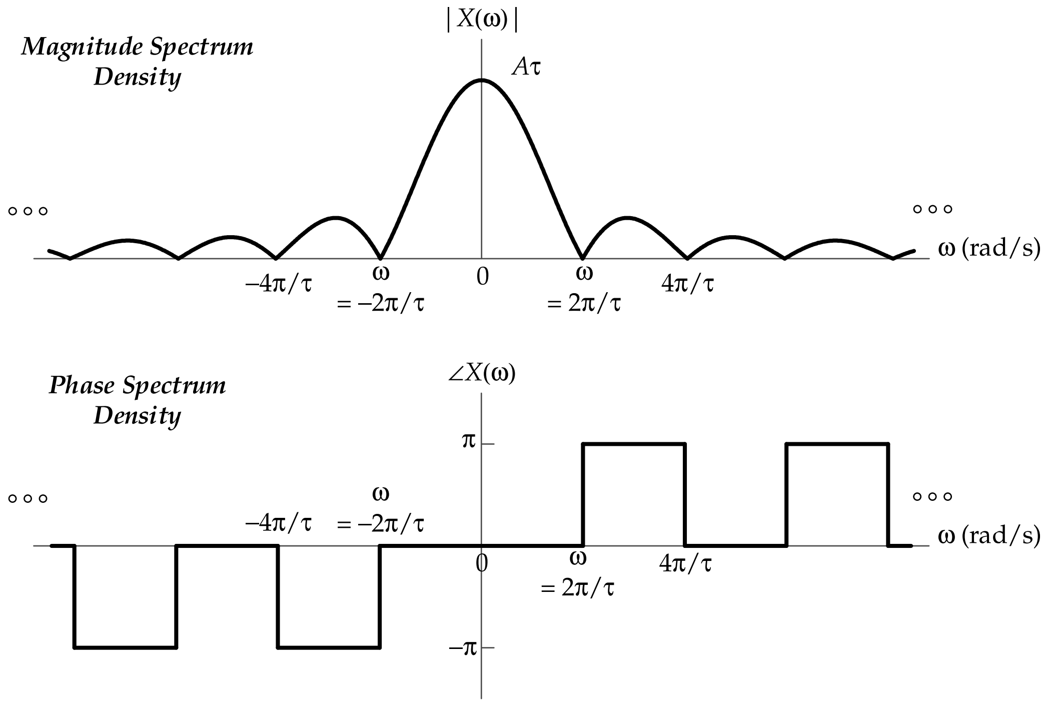
$$X(\omega) = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

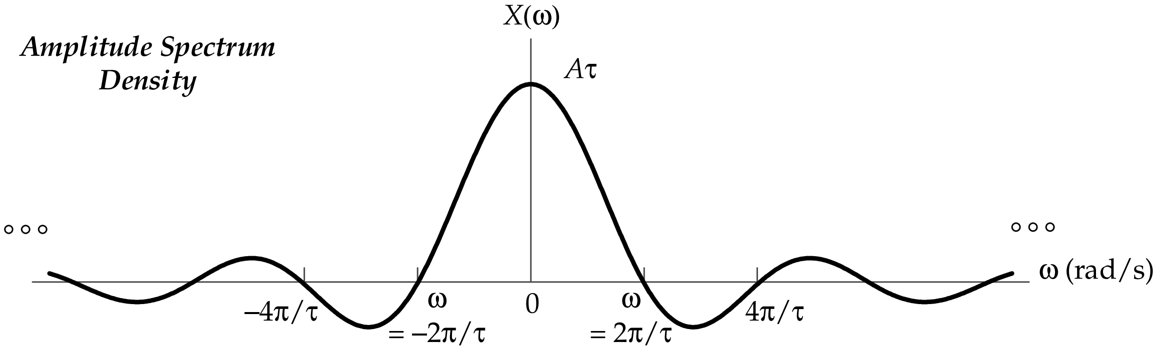
So,

$$|X(\omega)| = \left|A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)\right|$$

$$\angle X(\omega) = 0° \text{ or } 180°$$

Notice that $180° \equiv -180° \equiv \pi \equiv -\pi$.





Q. What is the bandwidth of $x(t)$?

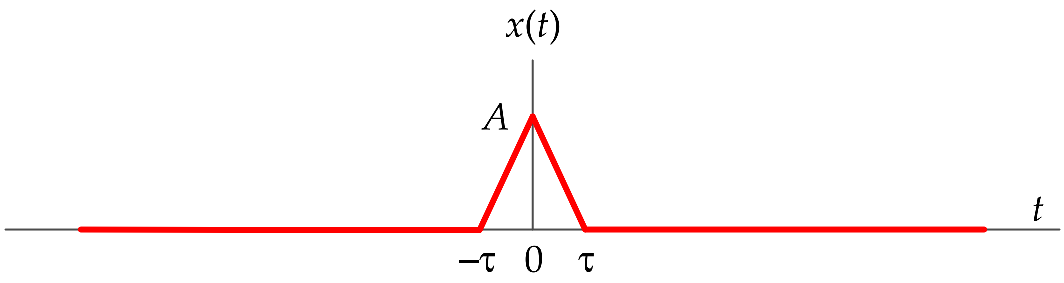
Compare the Fourier **transform** of the aperiodic $x(t) = A\ \mathrm{rect}\left(\frac{t}{\tau}\right)$, which is,

$$X(\omega) = A\tau\ \mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

to the Fourier **series** of the periodic $\mathrm{rep}_{T_0}\left\{A\ \mathrm{rect}\left(\frac{t}{\tau}\right)\right\}$, which is,

$$\alpha_n = \frac{A\tau}{T_0}\mathrm{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right), \quad \forall n$$

Q2. For the following signal $x(t) = A\,\Delta\left(\frac{t}{\tau}\right)$, determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q2. Answer. $X(\omega) = A\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$

Q2. Solution. Using Fourier transform integral

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(\omega) = \int_{-\infty}^{-\tau} 0\left(e^{-j\omega t}\right)dt + \int_{-\tau}^{0}\left(A + \frac{A}{\tau}t\right)\left(e^{-j\omega t}\right)dt + \int_{0}^{\tau}\left(A - \frac{A}{\tau}t\right)\left(e^{-j\omega t}\right)dt + \int_{\tau}^{\infty} 0\left(e^{-j\omega t}\right)dt$$

$$X(\omega) = A\int_{-\tau}^{0} e^{-j\omega t}dt + \frac{A}{\tau}\int_{-\tau}^{0}(t)\left(e^{-j\omega t}\right)dt + A\int_{0}^{\tau} e^{-j\omega t}dt + \frac{-A}{\tau}\int_{0}^{\tau}(t)\left(e^{-j\omega t}\right)dt$$

$$X(\omega) = A\int_{-\tau}^{0} e^{-j\omega t}dt + \frac{A}{\tau}\int_{-\tau}^{0}(t)\left(e^{-j\omega t}\right)dt$$
$$+ A\int_{0}^{\tau} e^{-j\omega t}dt + \frac{-A}{\tau}\int_{0}^{\tau}(t)\left(e^{-j\omega t}\right)dt$$
$$= A\left[\frac{1}{-j\omega}e^{-j\omega t}\right]_{-\tau}^{0} + \frac{A}{\tau}\times\frac{1}{(j\omega)^2}\left[(-j\omega t-1)e^{-j\omega t}\right]_{-\tau}^{0}$$
$$+ A\left[\frac{1}{-j\omega}e^{-j\omega t}\right]_{0}^{\tau} + \frac{-A}{\tau}\times\frac{1}{(j\omega)^2}\left[(-j\omega t-1)e^{-j\omega t}\right]_{0}^{\tau}$$

Note: Integration by parts $\int u\,dv = uv - \int v\,du$

$$\int (t)(e^{at})dt = e^{at}\left(\frac{at-1}{a^2}\right)$$

$$X(\omega) = A\left[\frac{1}{-j\omega}e^{-j\omega t}\right]_{-\tau}^{0} + \frac{A}{\tau}\times\frac{1}{(j\omega)^2}\left[(-j\omega t-1)e^{-j\omega t}\right]_{-\tau}^{0}$$
$$+ A\left[\frac{1}{-j\omega}e^{-j\omega t}\right]_{0}^{\tau} + \frac{-A}{\tau}\times\frac{1}{(j\omega)^2}\left[(-j\omega t-1)e^{-j\omega t}\right]_{0}^{\tau}$$
$$X(\omega) = \frac{A}{-j\omega}\left[1-e^{j\omega\tau}\right] + \frac{A}{\tau}\times\frac{1}{(j\omega)^2}\left[-1-(j\omega\tau-1)e^{j\omega\tau}\right]$$
$$+ \frac{A}{-j\omega}\left[e^{-j\omega\tau}-1\right] + \frac{-A}{\tau}\times\frac{1}{(j\omega)^2}\left[(-j\omega\tau-1)e^{-j\omega\tau}+1\right]$$
$$X(\omega) = \frac{A}{-j\omega}[1-1] + \frac{A}{-j\omega}\left[e^{-j\omega\tau}-e^{j\omega\tau}\right]$$
$$+ \frac{A}{\tau\,(j\omega)^2}\left[-1-1+(1-j\omega\tau)e^{j\omega\tau}+(1+j\omega\tau)e^{-j\omega\tau}\right]$$

$$X(\omega) = \frac{A}{-j\omega}[1-1] + \frac{A}{-j\omega}\left[e^{-j\omega\tau} - e^{j\omega\tau}\right]$$
$$+ \frac{A}{\tau\,(j\omega)^2}\left[-1-1+(1-j\omega\tau)e^{j\omega\tau} + (1+j\omega\tau)e^{-j\omega\tau}\right]$$

$$X(\omega) = \frac{A}{j\omega}\left[e^{j\omega\tau} - e^{-j\omega\tau}\right]$$
$$+ \frac{A}{\tau\,(j\omega)^2}\left[-1-1+(1-j\omega\tau)e^{j\omega\tau} + (1+j\omega\tau)e^{-j\omega\tau}\right]$$

$$X(\omega) = \frac{A}{(j\omega)^2}\left[j\omega e^{j\omega\tau} - j\omega e^{-j\omega\tau}\right]$$
$$+ \frac{A}{\tau\,(j\omega)^2}\left[-1-1+e^{j\omega\tau} - j\omega\tau e^{j\omega\tau} + e^{-j\omega\tau} + j\omega\tau e^{-j\omega\tau}\right]$$

$$X(\omega) = \frac{A}{(j\omega)^2}\left[j\omega e^{j\omega\tau} - j\omega e^{-j\omega\tau}\right]$$
$$+ \frac{A}{\tau\,(j\omega)^2}\left[-1-1+e^{j\omega\tau} - j\omega\tau e^{j\omega\tau} + e^{-j\omega\tau} + j\omega\tau e^{-j\omega\tau}\right]$$

$$X(\omega) = \frac{A}{\tau\,(j\omega)^2}\left[-1-1+e^{j\omega\tau} + e^{-j\omega\tau}\right]$$
$$= \frac{A\,e^{-j\omega\tau}}{\tau\,(j\omega)^2}\left[-e^{j\omega\tau} - e^{j\omega\tau} + e^{j2\omega\tau} + 1\right] = \frac{A\,e^{-j\omega\tau}}{\tau\,(j\omega)^2}\left[e^{j\omega\tau} - 1\right]^2$$

Because $(a-1)^2 = (a-1)(a-1) = a^2 - 2a + 1$

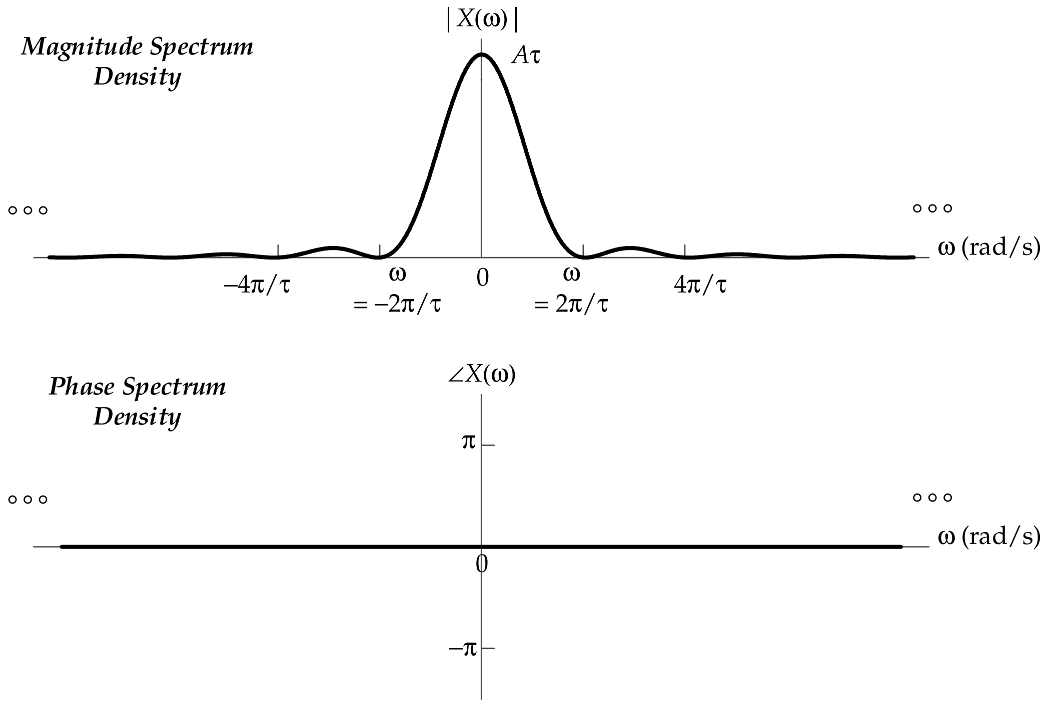
$$X(\omega) = \frac{A\,e^{-j\omega\tau}}{\tau\,(j\omega)^2}\left[e^{j\omega\tau} - 1\right]^2$$

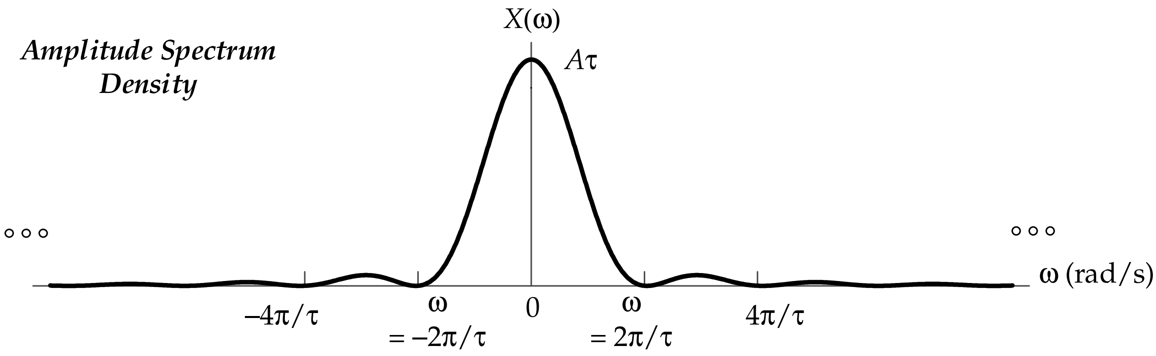
$$X(\omega) = \frac{A\,e^{-j\omega\tau}}{\tau\,(j\omega)^2}\left[e^{j\omega\tau/2}\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right)\right]^2$$

$$= \frac{A\,e^{-j\omega\tau}e^{j\omega\tau}}{\tau\,(j\omega)^2}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]^2 = \frac{A}{\tau\,(j\omega)^2}\left[2j\sin\left(\frac{\omega\tau}{2}\right)\right]^2$$

$$X(\omega) = \frac{A\,\tau^2}{\tau}\left[\frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}}\right]^2 = A\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

So, $|X(\omega)| = \left|A\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)\right| = A\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$ and $\angle X(\omega) = 0°$.





Q. What is the bandwidth of $x(t)$?

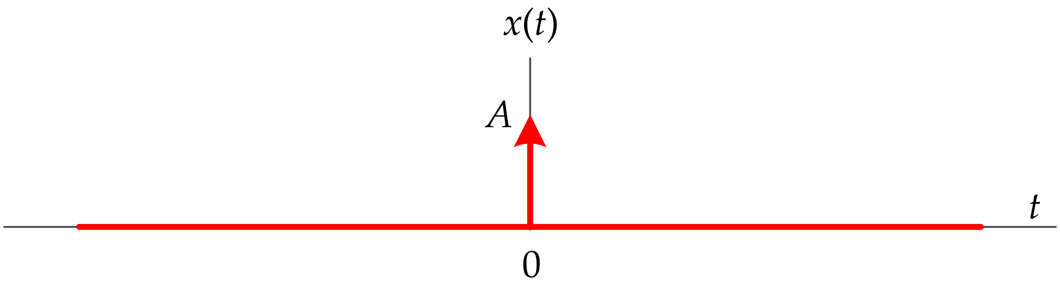
Compare the Fourier **transform** of the aperiodic $x(t) = A\,\Delta\left(\frac{t}{\tau}\right)$, which is,

$$X(\omega) = A\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

to the Fourier **series** of the periodic $\mathrm{rep}_{T_0}\left\{A\,\Delta\left(\frac{t}{\tau}\right)\right\}$, which is,

$$\alpha_n = \frac{A\tau}{T_0}\mathrm{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right), \quad \forall n$$

Q3. For the following signal $x(t) = A\,\delta(t)$ (impulse *or* Dirac delta signal), determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q3. Solution. Using Fourier transform integral

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

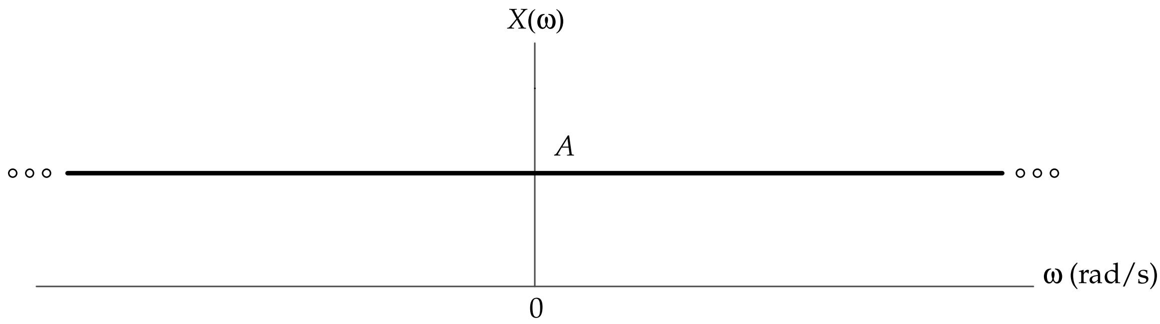
$$X(\omega) = \int_{-\infty}^{\infty} A\,\delta(t)\, e^{-j\omega t} dt = A \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt$$

Using the sampling property of the impulse

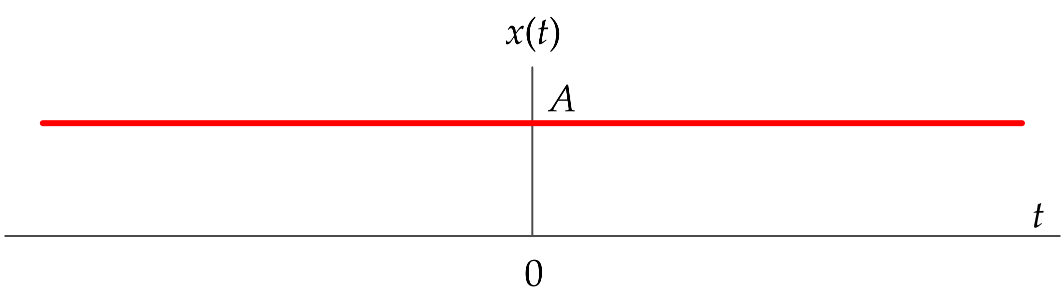
$$X(\omega) = A\, e^{-j\omega 0} = A$$

So, the Fourier transform of an impulse (located at time zero second) is constant in the frequency domain

$$X(\omega) = \mathcal{F}\{A\,\delta(t)\} = A$$



Q4. For the following signal $x(t) = A$ (i.e., constant or DC value or DC shift or offset), determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q4. Solution. We cannot use direct integration here because $x(t) = A$ is *not* absolutely integrable over the period $(-\infty, \infty)$. In other words, it does *not* satisfy $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$. However, since the answer turns out to be a simple one, we can guess it, and verify that it is the correct answer by performing an inverse Fourier transform. Assuming

$$X(\omega) = 2\pi A\, \delta(\omega)$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi A\, \delta(\omega)\, e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi A\, \delta(\omega)\, e^{j\omega t} d\omega = A \int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t} d\omega$$

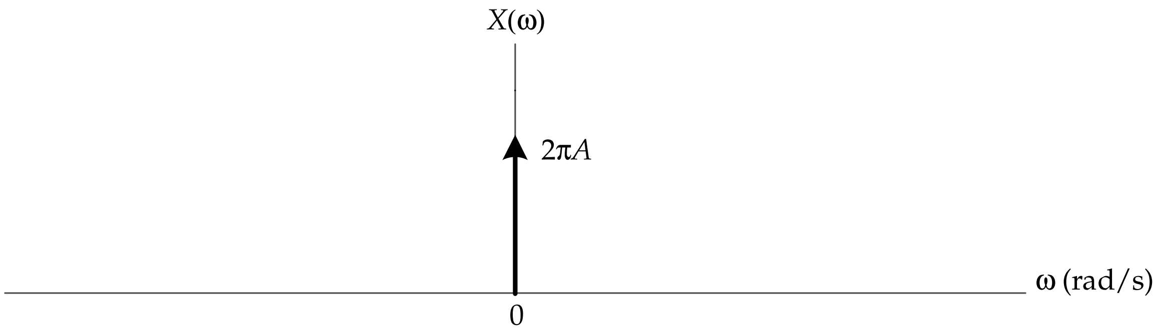
Using the sampling property of the impulse (located at $\omega = 0$)

$$x(t) = A\, e^{j(0)t} = A$$

Hence,

$$X(\omega) = \mathcal{F}\{A\} = 2\pi A\, \delta(\omega)$$

So, the Fourier transform of a DC shift or DC offset is one impulse around zero multiplied by 2π. This makes sense because this is just the α_0 multiplied by 2π and converted into an impulse.



Q5. For the following Fourier transform $X(\omega) = 2\pi A\, \delta(\omega - \omega_0)$ (shifted impulse), determine the inverse Fourier transform $x(t) = \mathcal{F}^{-1}\{X(\omega)\}$.

Q5. Answer.

$$x(t) = Ae^{j\omega_0 t}$$

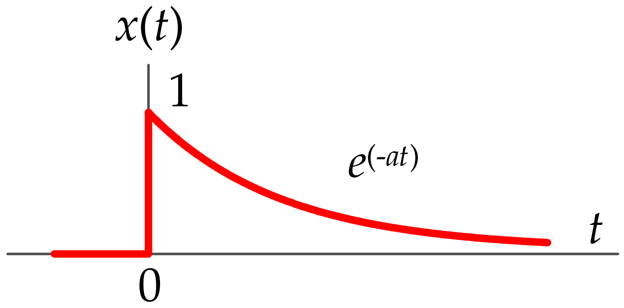
In other words,

$$X(\omega) = \mathcal{F}\left\{x(t) = Ae^{j\omega_0 t}\right\} = 2\pi A\, \delta(\omega - \omega_0)$$

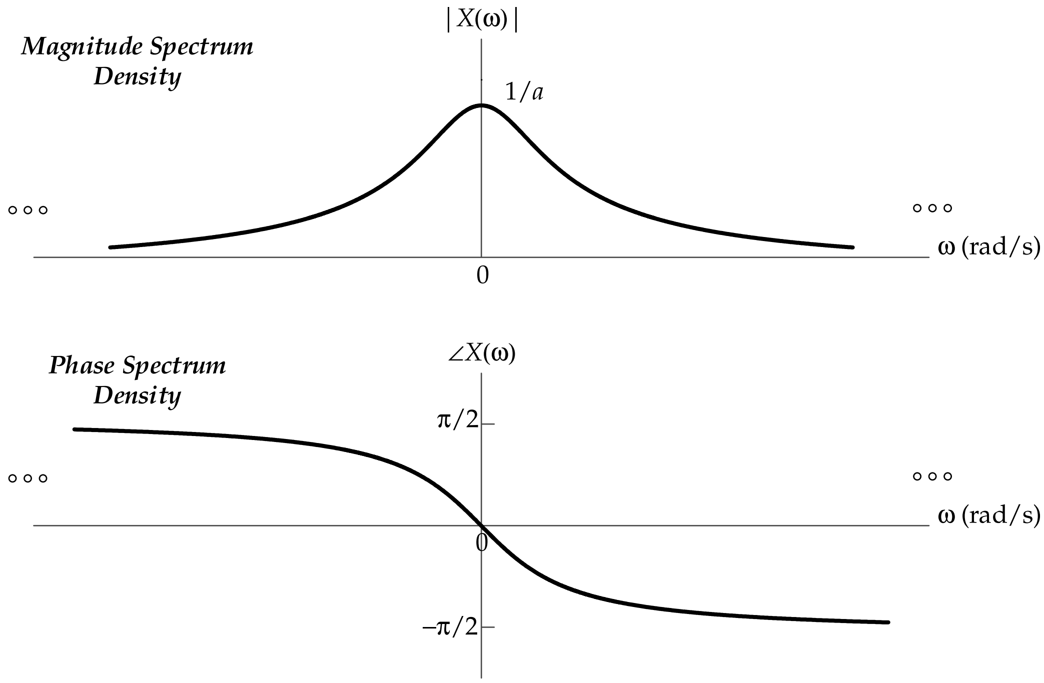
Also, you can prove that

$$X(\omega) = \mathcal{F}\left\{x(t) = Ae^{-j\omega_0 t}\right\} = 2\pi A\, \delta(\omega + \omega_0)$$

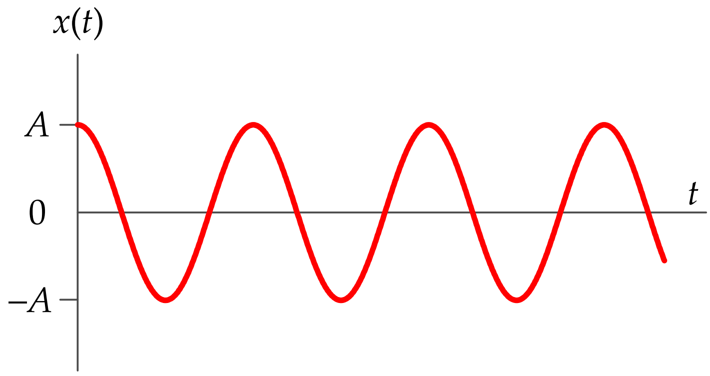
Q6. For the following signal $x(t) = e^{-at}u(t)$, $a > 0$, determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q6. Answer. $X(\omega) = \frac{1}{a+j\omega}$, $|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$, $\angle X(\omega) = -\tan^{-1}(\omega/a)$



Q7. For the following signal $x(t) = A\cos(\omega_0 t)$, determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q7. Solution #1. This is a periodic signal, so we can find the complex exponential Fourier series (by inspection)

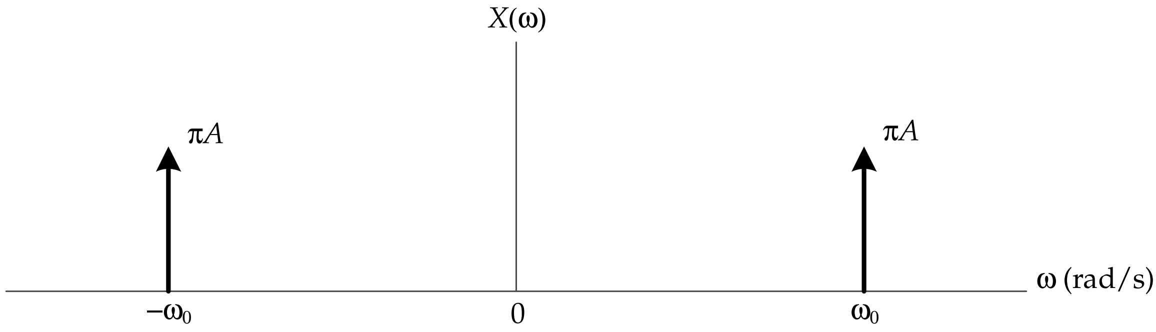
$$x(t) = A\cos(\omega_0 t) = A\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

The result is: $\alpha_0 = 0$, $\alpha_1 = A/2$, $\alpha_{-1} = A/2$, otherwise $\alpha_n = 0$.

Now to convert to Fourier transform: Multiply each α_n by 2π, and convert values α_n into impulses (at proper frequencies). This gives,

$$X(\omega) = \mathcal{F}\{x(t)\} = 2\pi\frac{A}{2}\,\delta(\omega - \omega_0) + 2\pi\frac{A}{2}\,\delta(\omega + \omega_0)$$

$$\mathcal{F}\{A\cos(\omega_0 t)\} = \pi A\,\delta(\omega - \omega_0) + \pi A\,\delta(\omega + \omega_0)$$



Q7. Solution #2. Using Fourier transform integral

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(\omega) = \int_{-\infty}^{\infty} A\cos(\omega_0 t)\, e^{-j\omega t}dt$$

$$X(\omega) = \int_{-\infty}^{\infty} A\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-j\omega t}dt$$

$$X(\omega) = \frac{A}{2}\int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt + \frac{A}{2}\int_{-\infty}^{\infty} e^{-j\omega_0 t}e^{-j\omega t}dt$$

$$X(\omega) = \frac{A}{2}\mathcal{F}\left\{e^{j\omega_0 t}\right\} + \frac{A}{2}\mathcal{F}\left\{e^{-j\omega_0 t}\right\}$$

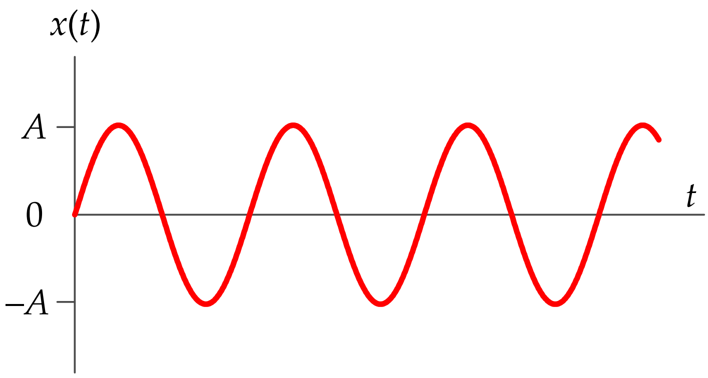
Using the result of our earlier example,

$$X(\omega) = \frac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\} + \frac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}$$

$$X(\omega) = \frac{A}{2} \times 2\pi\ \delta(\omega - \omega_0) + \frac{A}{2} \times 2\pi\ \delta(\omega + \omega_0)$$

$$X(\omega) = \pi A\ \delta(\omega - \omega_0) + \pi A\ \delta(\omega + \omega_0)$$

Q8. For the following signal $x(t) = A\sin(\omega_0 t)$, determine the Fourier transform $X(\omega) = \mathcal{F}\{x(t)\}$ and sketch the corresponding magnitude and phase spectrum densities.



Q8. Answer. $X(\omega) = -j\pi A\ \delta(\omega - \omega_0) + j\pi A\ \delta(\omega + \omega_0)$